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EUCLIDEAN n-SPACE MODULO AN $(n-1)$-CELL[1]

BY
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ABSTRACT. This paper, together with another paper by the author titled similarly, provides a complete answer to a conjecture raised by Andrews and Curtis: *if D is a k-cell topologically embedded in euclidean n-space E^n, then $E^n/D \times E^1$ is homeomorphic to E^{n+1}.* Although there is at present only one case outstanding ($n \geqslant 4$ and $k = n-1$), the proof we give here works whenever $n \geqslant 4$. We resolve this conjecture (for $n \geqslant 4$) by proving a stronger result: *if $Y \times E^1 \approx E^{n+1}$ and if D is a k-cell in Y, then $Y/D \times E^1 \approx E^{n+1}$.* This theorem was proved by Glaser for $k \leqslant n-2$ and has as a corollary: *if K is a collapsible polyhedron topologically embedded in E^n, then $E^n/K \times E^1 \approx E^{n+1}$.* Our method of proof uses radial engulfing and a well-known procedure devised by Bing.

1. **Introduction.** In [1] Andrews and Curtis proved that if A is an arc in euclidean n-space E^n, then $E^n/A \times E^1$ is homeomorphic to E^{n+1}. They conjectured that a similar phenomenon occurs for a k-cell D topologically embedded in E^n. In [4] the author proved that $E^n/D \times E^1 \approx E^{n+1}$ whenever D is flat in E^{n+1}. This condition is known to be satisfied except (possibly) when $n \geqslant 4$ and $k = n-1$. (See [11], [7], [5], and [6].)

The main result of this paper is that $E^n/D \times E^1 \approx E^{n+1}$ in the one situation that remains ($n \geqslant 4$ and $k = n-1$). The proof we give actually works for any $k = 1, 2, \ldots, n$ so long as $n \geqslant 4$. It uses a generous application of the engulfing theorems of Bing [3], Seebeck [13], and Wright [16] and the methods of [1], [2], and [4]. It has the added feature that it does not involve a higher dimensional PL (or locally flat) approximation theorem for cells—either directly or indirectly. Thus, combining [4] and the present paper we obtain that, in general, *if D is a k-cell topologically embedded in E^n, then $E^n/D \times E^1 \approx E^{n+1}$.*

The theorem we shall prove is a generalization of this statement in the case $n \geqslant 4$.

Theorem 1.1. *Suppose that Y is a space with the property that $Y \times E^1 \approx E^{n+1}$ ($n \geqslant 4$) and that D is a k-cell topologically embedded in Y. Then $Y/D \times E^1 \approx E^{n+1}$.*

This generalization has been proved by Glaser in case $n = 3$ or $n \geqslant 4$ and $k \leqslant n-2$ [8]. (For $n = 3$, one must also assume, however, that D is flat in E^4.) Its importance can be seen from the following corollary.

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Corollary (Glaser, [8]). *If K is a collapsible polyhedron topologically embedded in E^n, then $E^n/K \times E^1 \approx E^{n+1}$.*

Many thanks go to the referee for discovering a serious error in the engulfing theorem of the original version of this paper. The correcting of this mistake led to a considerable simplification of the original manuscript.

2. **Definitions and notation.** We use "$\sim$", "$\simeq$", and "$\approx$" to mean "is homologous to" (integer coefficients), "is homotopic to," and "is homeomorphic to," respectively. Let G be a subset of a metric space X. The ϵ-neighborhood of a point $x \in X$ is denoted by $N_\epsilon(x)$. We say that G is p-lc (p-LC) at a point $x \in X$ iff for each $\epsilon > 0$ there exists $\delta > 0$ such that every singular p-cycle (p-sphere) in $N_\delta(x) \cap G$ is homologous to zero (homotopic to zero) in $N_\epsilon(x) \cap G$. The set G is lc^p (LC^p) at $x \in X$ iff G is q-lc (q-LC) at x for $0 \leqslant q \leqslant p$. G is lc^∞ (LC^∞) at x iff G is q-lc (q-LC) for all $q \geqslant 0$. The terms p-ulc, p-ULC, ulc^p, and ULC^p are used whenever the δ corresponding to ϵ and x above may be chosen independently of x. Finally, we say that G has property 1-ALG at $x \in X$ [9] iff for each $\epsilon > 0$ there exists $\delta > 0$ such that for each singular 1-sphere Γ in $N_\delta(x) \cap G$, $\Gamma \sim 0$ in $N_\epsilon(x) \cap G$ iff $\Gamma \simeq 0$ in $N_\epsilon(x) \cap G$.

We use I to denote the unit interval $[0, 1]$ and I^k to denote the cartesian product $I \times \ldots \times I$, k factors. Suppose that $Y \times E^1 \approx E^{n+1}$. The metric we shall use on $Y \times E^1$ is the metric d defined by

$$d((y,t),(y',t')) = \max\{\|(y,0) - (y',0)\|, |t - t'|\},$$

where $\|\cdot\|$ is the usual norm on E^{n+1}. The projections of E^{n+1} onto Y and E^1 are denoted by π_1 and π_2, respectively.

3. **$E^{n+1} - D$ is 1-ALG at points of D.** Throughout this section Y is a space such that $Y \times E^1 \approx E^{n+1}$ and D is a k-cell topologically embedded in Y ($= Y \times 0$). In the original version of this paper we showed that the embedding of D into E^{n+1} has a much stronger property than that stated in the title of this section; namely, that D is *locally homotopically unknotted* [6] in E^{n+1}. Our present proof of Theorem 1.1, however, does not require anything this strong.

Theorem 3.1. *$E^{n+1} - D$ is LC^1 at each point of* Bd D.

Proof. Notice that $E^{n+1} - D$ is lc^∞ at each point of Bd D by local duality [15]; hence, we need only show that $E^{n+1} - D$ is 1-LC at all such points. Also, $E^{n+1} - (D \times E^1)$ is lc^∞ at points of Bd $D \times E^1$, which implies that $Y - D$ is lc^∞ at points of Bd D.

Suppose that $y \in$ Bd D and $\epsilon > 0$. Choose $\gamma > 0$ and $\delta > 0$ so that $N_\gamma(y) \cap Y$ is contractible to a point in $N_\epsilon(y) \cap Y$ and each pair of points in $N_\delta(y) \cap (Y - D)$ can be joined by a path in $N_\gamma(y) \cap (Y - D)$. (Y is locally contractible since it is a retract of E^{n+1}.)

Let $f: S^1 \to (N_\delta(y) - D)$ be given. Choose $\eta > 0$ so that if $g: S^1 \to E^{n+1}$ is a map with $d(f,g) < \eta$, then $f \simeq g$ in $N_\delta(y) - D$. Let T be a subdivision of S^1 with the following properties:

(i) diam $f(A) < \eta/2$ for all $A \in T$,

(ii) if $f(A) \cap Y \neq \varnothing$, then $\pi_1 f(A) \subset N_\delta(y) \cap (Y - D)$,

(iii) if v is a vertex in T, then $f(v) \notin Y$. (We may have to change f by a small homotopy in order to satisfy this condition.)

Let A be a 1-simplex of T such that $f(A) \cap Y \neq \varnothing$. Parameterize A by t, $0 \leqslant t \leqslant 1$, and define $g_A : A \to E^{n+1}$ by

$$g_A(t) = (\pi_1 f(t), (1-t)\pi_2 f(0) + t\pi_2 f(1)).$$

Observe that $g_A(0) = f(0)$, $g_A(1) = f(1)$ and $d(g_A(t), f(t)) \leqslant d(g_A(t), g_A(0)) + d(f(0), f(t)) < \eta$ for each $t \in I$. Define $g: S^1 \to E^{n+1}$ by

$$\begin{aligned} g(x) &= g_A(x), && \text{if } x \in A \text{ and } f(A) \cap Y \neq \varnothing, \\ &= f(x), && \text{otherwise.} \end{aligned}$$

Then $d(f,g) < \eta$ and so $f \simeq g$ in $N_\delta(y) - D$. Observe also that $g(S^1) \cap Y$ is a finite set.

Let α be a subarc of $g(S^1)$ joining successive points a and b of $g(S^1) \cap Y$. By our choice of δ, a and b can be joined by a path β in $N_\gamma(y) \cap (Y - D)$. By our choice of γ, $\beta \cup \pi_1(\alpha)$ bounds a singular disk Δ in $N_\epsilon(y) \cap Y$. Let $v * S^1 = \{tv + (1-t)w \mid w \in S^1, 0 \leqslant t \leqslant 1\}$ be the (abstract) cone over S^1, and let $g' : v * S^1 \to \Delta$ be a map with $g'(S^1) = \beta \cup \pi_1(\alpha)$. Let $h : S^1 \to \alpha \cup \beta$ be a map such that $\pi_1 h = g' \mid S^1$. Extend h to a map of $v * S^1$ into E^{n+1} by the formula

$$h(tv + (1-t)w) = (g'(tv + (1-t)w), t\epsilon' + (1-t)\pi_2 h(w))$$

for $0 \leqslant t \leqslant 1$, $w \in S^1$, where $\epsilon' = \pm\epsilon$ accordingly as Int $\alpha \subset Y \times (0,\infty)$ or Int $\alpha \subset Y \times (-\infty, 0)$.

Then $h(v * S^1) \cap Y = \beta$ yields $h(v * S^1) \subset N_\epsilon(y) - D$. Applying this procedure to each arc α in $g(S^1)$ joining successive points of $g(S^1) \cap Y$, we obtain a homotopy of g in $N_\epsilon(y) - D$ to a map $g_1 : S^1 \to N_\gamma(y) \cap (Y - D)$. Next homotope g_1 to $g_2 : S^1 \to N_\gamma(y) \cap (Y \times \{\epsilon/2\})$ by pushing along the E^1 factor of $Y \times E^1$. Again by our choice of γ, g_2 is null-homotopic in $N_\epsilon(y) \cap (Y \times \{\epsilon/2\})$. Piecing these homotopies together, we get a homotopy of g to 0 in $N_\epsilon(y) - D$; hence, $f \simeq 0$ in $N_\epsilon(y) - D$.

Corollary 3.2. *If $k \neq n-1$, then $E^{n+1} - D$ is 1-LC at each point of D.*

Proof. For $k < n-1$ this follows from the proof of Theorem 3.1, since local duality implies that $Y - D$ is 0-lc at each point of D. If $k = n$, then the fact that Y is locally contractible implies that $E^{n+1} - D$ is 1-LC at points of Int D. (In fact, D is locally flat at points of Int D.)

Theorem 3.3. *If* $k = n - 1$, $E^{n+1} - D$ *is* 1-*ALG at points of* Int D.

Proof. We proceed very much the same as in the proof of Theorem 3.1. Suppose that $y \in \text{Int } D$ and $\epsilon > 0$. Choose $\gamma > 0$ so that $N_\gamma(y) \cap Y$ is contractible to a point in $N_\epsilon(y) \cap Y$. Since $D \times E^1$ is locally 2-sided in E^{n+1} at y, D is locally 2-sided in Y at y. Thus there is a neighborhood U of y in E^{n+1} lying in $N_\gamma(y)$ such that $U \cap (Y - D)$ has exactly two components (which are separated in $N_\epsilon(y) \cap (Y - D)$). Choose $\delta > 0$ so that $N_\delta(y) \subset U$. Let Γ' be a simple closed curve in $N_\delta(y) - D$ that is homologous to zero in $N_\epsilon(y) - D$. From the proof of Theorem 3.1, we see that Γ' is homotopic (in $N_\delta(y) - D$) to a simple closed curve Γ such that Γ meets Y "transversally" in a finite number of points —that number necessarily being an even integer.

Write $\Gamma \cap Y = \{x_0, x_1, \ldots, x_{2m-1}\}$, where the x_i's are arranged cyclically on Γ. Let

$$j : H_1(N_\epsilon(y) - D) \to H_0(N_\epsilon(y) \cap (Y - D))$$

be the homomorphism obtained from the Mayer-Vietoris sequence of the triad $(N_\epsilon(y) - D; N_\epsilon(y) \cap ((Y \times [0, \infty)) - D), N_\epsilon(y) \cap ((Y \times (-\infty, 0]) - D))$, and let

$$i : H_1(N_\delta(y) - D) \to H_1(N_\epsilon(y) - D)$$

be the inclusion induced homomorphism. Then

$$ji([\Gamma]) = \sum_{r=0}^{2m-1} (-1)^{r+1}[x_r] = 0.$$

Since $\{x_0, x_1, \ldots, x_{2m-1}\}$ lies in the union of two components of $N_\epsilon(y) \cap (Y - D)$, it must be true that for some r (mod $2m$), x_r and x_{r+1} lie in the same component of $N_\epsilon(y) \cap (Y - D)$; hence, in the same component of $U \cap (Y - D)$. An argument similar to one given in the proof of Theorem 3.1 can now be used to show that Γ is homotopic (in $N_\epsilon(y) - D$) to a simple closed curve Γ_1 in $U - D$ that has two fewer intersections with Y. Applying this process m times, we arrive at a curve Γ_m in $U - D$, homotopic to Γ in $N_\epsilon(y) - D$, such that $\Gamma \cap Y = \varnothing$. Thus, by the choice of γ, $\Gamma \simeq 0$ in $N_\epsilon(y) - D$, and we are through.

4. **Engulfing.** Throughout this section we shall use the following notation:
$f : I^{k-1} \times I \to E^m$ is an embedding ($m \geqslant 5, k < m$),
$D[a,b] = f(I^{k-1} \times [a,b])$,
$D[a] = D[a,a]$, and
$E^m - D[a,b]$ is 1-ALG at each point of $D[a,b]$ for $0 \leqslant a \leqslant b \leqslant 1$.

Proposition 4.1. *Suppose that* W *is a neighborhood of* $D[0,a]$ *and* $\epsilon > 0$. *Then there exist* $\delta > 0$ *and a neighborhood* W' of $D[0,a]$ *with the following properties: If* P *is an* $(m - 3)$*-polyhedron in* $N_\delta(D[a,b])$, *then there exists an isotopy* h_t $(t \in I)$ *of* E^m *such that*

(i) $h_0 =$ *identity*,

(ii) $h_t =$ *identity on* W' *and outside* $N_\epsilon(D[a,b])$,

(iii) $h_1(W) \supset P$, *and*

(iv) *if* $z \in E^m$, *either* $h_t(z) = z$ *for all* $t \in I$ *or there exists* $x \in I^{k-1}$ *such that* $h_t(z) \in N_\epsilon(f(x \times [a,b]))$ *for all* $t \in I$.

Proof. Suppose that W is a neighborhood of $D[0,a]$ and $\epsilon > 0$. We shall construct the homotopies necessary to apply radial engulfing.

Let $r_t : E^m \to E^m$ be the "straight-line" homotopy between the identity (r_0) and a retraction (r_1) of E^m onto $D[a,b]$. Choose $c > a$ so that $D[0,c] \subset W$. Then there exist neighborhoods U of $D[0,a]$ and V of $D[c,b]$ such that $r_s(\overline{U}) \cap r_t(\overline{V}) = \varnothing$ for all $s, t \in I$. Let V' be a neighborhood of $D[c,b]$ such that $\overline{V'} \subset V$ and let $\alpha : E^m \to I$ be a mapping such that $\alpha(E^m - V) = 0$ and $\alpha(\overline{V'}) = 1$.

Define $\phi_t : E^m \to E^m$ by $\phi_t(y) = r_{t\alpha(y)}(y)$. Then $\phi_0 =$ identity, $\phi_t(y) \in U$ for some $t \in I$ implies $\phi_t(y) = y$ for all $t \in I$, $\phi_1 | V' : V' \to V' \cap D[a,b]$ is a retraction and $\phi_t | D[0,b] =$ identity for all $t \in I$.

Let ψ_t be the natural homotopy of the identity on $D[c,b]$ to the projection of $D[c,b]$ onto $D[c]$.

The homotopy ϕ_t followed by ψ_t will pull all sufficiently small neighborhoods of $D[a,b]$ into W. Moreover, for every $\delta > 0$ there is a neighborhood N of $D[a,b]$ such that $\phi_t | N$ is a δ-homotopy. Thus, the engulfing techniques of [3], [13], and [16] can be applied to give the desired isotopies.

An important observation is that the neighborhood W' of $D[0,a]$ depends only upon W and the embedding f (or, more precisely, the deformation retraction r_t).

Lemma 4.2. *Suppose that* K *is a* 2*-complex in* E^m *such that* $K \cap D[0,a] = \varnothing$. *Then for each* $\epsilon > 0$ *there exists a homotopy* $g_t : K \to E^m$ ($t \in I$) *such that*

(1) $g_0 =$ *inclusion*,

(2) $g_t | K - N_\epsilon(D[a,b]) =$ *inclusion*,

(3) $g_t(K) \cap D[0,a] = \varnothing$ *for each* $t \in I$,

(4) $g_1(K) \cap D[0,b] = \varnothing$, *and*

(5) *for each* $y \in K$ *either* $g_t(y) = y$ *for all* $t \in I$ *or there exists* $x \in I^{k-1}$ *such that* $g_t(y) \in N_\epsilon(f(x \times [a,b]))$.

Proof. *Case* 1. $k \leqslant m - 3$. This situation is easy to handle since $E^m - D$ is 1-ULC.

Case 2. $k = m - 2$. Let T be a fine triangulation of K. Since $m \geqslant 5$ and $k = m - 2$, the 1-skeleton T^1 of T can be moved off of $D[a,b]$ with an arbitrarily small isotopy of E^m that is fixed outside a neighborhood of $D[a,b]$. So we will assume that $T^1 \cap D[a,b] = \varnothing$. Let σ be a 2-simplex of T that meets $D[a,b]$, and let v be a point of $\sigma \cap D[a,b]$ ($v \in \operatorname{Int} \sigma$). Then $\sigma = \{sv + (1-s)z \mid s \in I, z \in \operatorname{Bd} \sigma\}$. Given $0 \leqslant r \leqslant s \leqslant 1$, define

$$C(r) = \{rv + (1-r)z \mid z \in \operatorname{Bd} \sigma\}$$

and

$$C(r,s) = \{tv + (1 - t)z \mid r \leqslant t \leqslant s,\ z \in \text{Bd } \sigma\}.$$

Choose $s_0, s_1 \in I$ so that $0 < s_0 < s_1 < 1$ and $C(0,s_1) \cap D[a,b] = \varnothing$. Write $v = f(x,c)$, where $x \in I^{k-1}$ and $a < c \leqslant b$. (See Figure 1.)

Let $\alpha_t : [s_0, 1] \to (0, \infty)$ $(t \in I)$ be the linear map satisfying $\alpha_t(s_0) = s_0$ and $\alpha_t(s_1) = (1 - t)s_1 + t$, and let $\beta(s,t) \in [c,b]$ $(s,t \in I)$ satisfy the equation

$$\frac{\beta(s,t) - c}{[(1 - t)c + tb] - c} = \frac{s - \alpha_t^{-1}(t)}{1 - \alpha_t^{-1}(1)}$$

whenever $t > 0$. (Note, $\alpha_0^{-1}(1) = 1$.)

Define $g'_t : \sigma \to E^m$ $(t \in I)$ by

$$\begin{aligned} g'_t(sv + (1 - s)z) &= sv + (1 - s)z && \text{if } 0 \leqslant s \leqslant s_0 \text{ or } t = 0, \\ &= \alpha_t(s)v + (1 - \alpha_t(s))z && \text{if } s_0 \leqslant s \leqslant \alpha_t^{-1}(1)\ (t > 0), \\ &= f(x, \beta(s,t)) && \text{if } \alpha_t^{-1}(1) \leqslant s \leqslant 1\ (t > 0). \end{aligned}$$

Then g'_t has the property that g'_0 = inclusion, $g'_t \mid C(0,s_0)$ = inclusion, $g'_1(C(s_0,s_1)) = C(s_0,1)$, and $g'_1(C(s)) = f(x,\beta(s,1))$ for $s_1 \leqslant s \leqslant 1$.

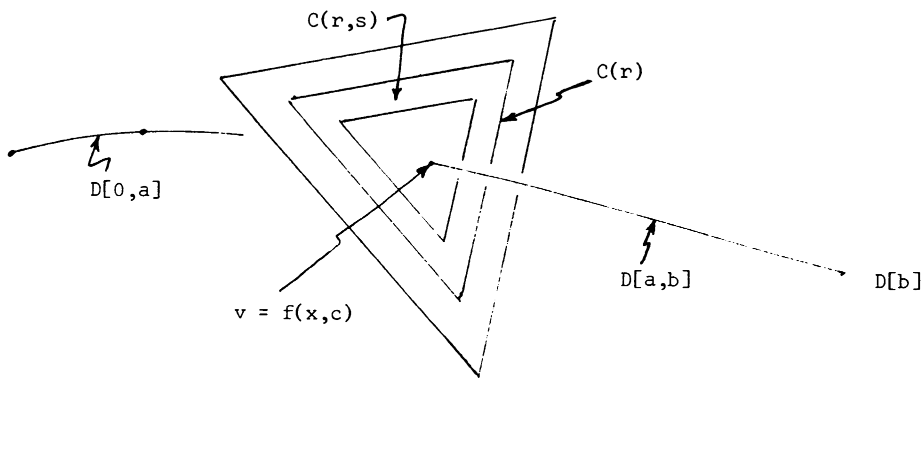

Figure 1

From local duality [15] we know that if $a < d \leqslant b$ and U is a neighborhood of $f(x,d)$ in E^m, then there exists a neighborhood V of $f(x,d)$ in U such that the image of the inclusion $i_* : H_1(V - D[a,b]) \to H_1(U - D[a,b])$ is either zero (if $x \in \text{Bd } I^{k-1}$ or $d = b$) or possibly the integers (if $(x,d) \in \text{Int } D[a,b]$). Hence, if y is a point of $V \cap f(x \times [a,b])$ and if $z \in \text{im } i_*$, then arbitrarily close to y there is a simple closed curve Γ in $V - D[a,b]$ such that $i_*([\Gamma]) = z$. Therefore, assuming that the diameter of σ is sufficiently small, we can use standard compactness arguments to find numbers $s_2, \ldots, s_n \in [s_1, 1]$ with $s_1 < s_2 < \ldots < s_n < 1$ and simple closed curves $\Gamma_1, \ldots, \Gamma_n$ such that (taking $\Gamma_0 = C(s_0)$)

$$\Gamma_i \subset (\text{neighborhood of } g_1'(C(s_i))) - D[a,b],$$

$$\Gamma_i \sim \Gamma_{i-1} \quad \text{in (larger neighborhood of } g_1'(C(s_i))) - D[a,b], \text{ and}$$

$$\Gamma_n \sim 0 \quad \text{in (neighborhood of } f(x,b)) - D[a,b]. \text{ (See Figure 2.)}$$

Using the 1-ALG property of $E^m - D[a,b]$, we see that if $\Gamma_1, \ldots, \Gamma_n$ are suitably chosen, then $\Gamma_i \cup \Gamma_{i-1}$ $(i = 1, \ldots, n)$ bounds a singular annulus in $N_{\epsilon/2}(g_1'(C(s_i))) - D[a,b]$ and Γ_n bounds a singular disk in $N_{\epsilon/2}(f(x,b)) - D[a,b]$. Thus, we can find a map $g : \sigma \to N_{\epsilon/2}(f(x \times [a,b])) - D[a,b]$ such that $g(\sigma) \cap D[0,a] = \varnothing$, $g \mid \mathrm{Bd}\, \sigma =$ inclusion, and g is $(\epsilon/2)$-homotopic (rel Bd σ) to g_1'. Piecing these homotopies together as σ ranges over the simplexes of K that meet $D[a,b]$ gives the desired homotopy g_t $(t \in I)$ of K in E^m.

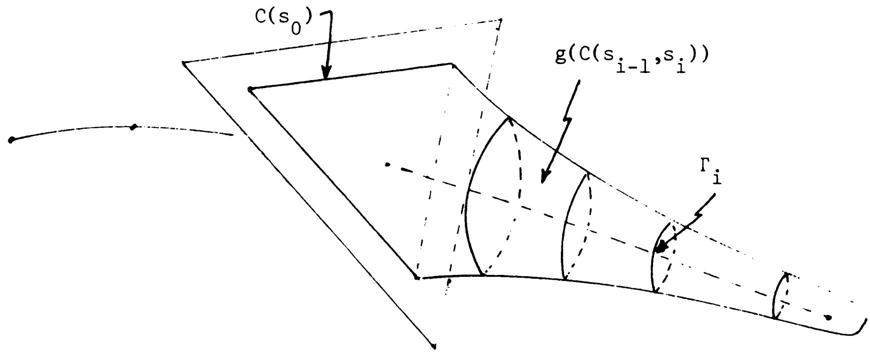

Figure 2

Case 3. $k = m - 1$. Again let T be a fine triangulation of K. We shall assume that no vertex of T lies in $D[a,b]$ and that no 1-simplex of T meets Bd $D[a,b]$. We shall also assume that no 1-simplex of T that meets $D[a,b]$ has both of its vertices "on the same side" of $D[a,b]$. Thus, every 2-simplex of T meets $D[a,b]$ in essentially one of two ways as illustrated in Figure 3.

We proceed in much the same way as in Case 2. Let e be a 1-simplex of T that meets $D[a,b]$, and let v be a point of intersection. Write $v = f(x,c)$, where $x \in I^{k-1}$ and $a < c < b$. Given $0 \leqslant r \leqslant s \leqslant 1$, define

$$A(r) = \{rv + (1-r)z \mid z \in \mathrm{Bd}\, e\}$$

and

$$A(r,s) = \{tv + (1-t)z \mid z \in \mathrm{Bd}\, e, r \leqslant t \leqslant s\}.$$

Given a 2-simplex σ of T that meets $D[a,b]$ and $0 \leqslant r \leqslant s \leqslant 1$, define subsets $C(r)$ and $C(r,s)$ of σ as in Figure 3. The specific formula for $C(r)$ (and $C(r,s)$) will, of course, depend on whether one or two of the edges of σ meet $D[a,b]$. Choose s_0 and s_1 so that $0 < s_0 < s_1 < 1$ and $C(0,s_1) \cap D[a,b] = \varnothing$ for the set $C(0,s_1)$ corresponding to each 2-simplex σ of T that meets $D[a,b]$. If a 2-simplex σ of T has two of its edges intersecting $D[a,b]$, we homotope σ (rel Bd σ) so that the segment $C(1)$ is carried "linearly" onto the image under f of the segment in I^k joining (x_1,c_1) and (x_2,c_2), where $v_1 = f(x_1,c_1)$ and $v_2 = f(x_2,c_2)$ are the distinquished points Bd $\sigma \cap D[a,b]$.

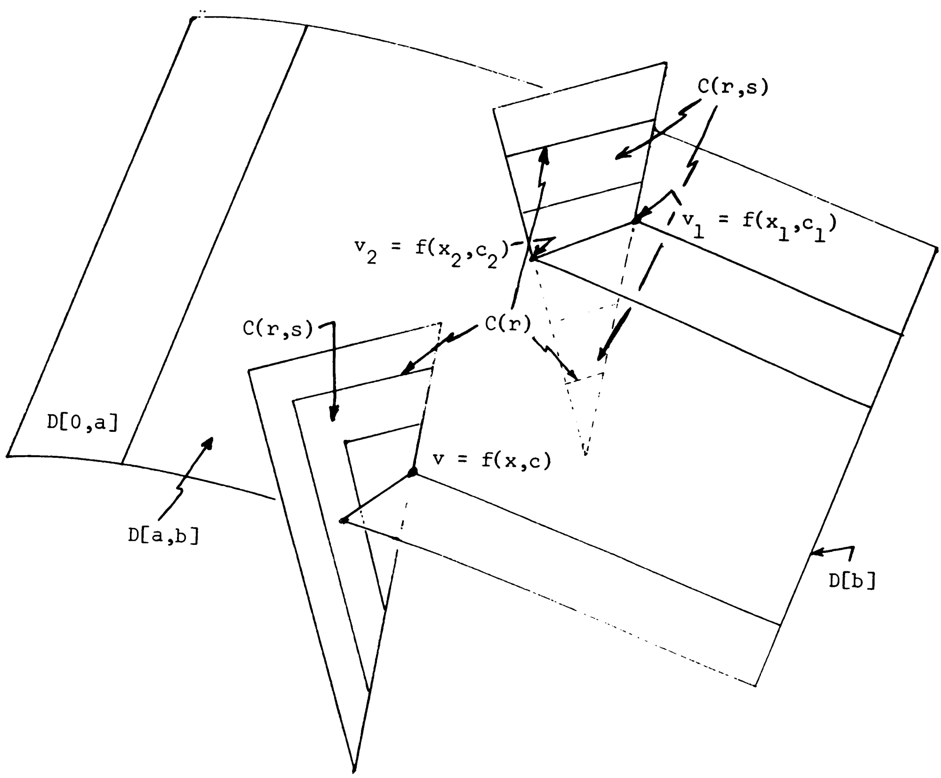

Figure 3

Next, construct $g'_t : T \to E^n$ $(t \in I)$ using the same type of formulas as in Case 2 on each 2-simplex of T that meets $D[a,b]$. Since $E^m - D[a,b]$ is 1-ULC, the map $g'_1 : T \to E^m$ will be homotopic, via a small homotopy, to a map $g : T \to E^m - D[a,b]$. (See Figure 4.) The combination of the three homotopies is the desired homotopy f_t $(t \in I)$.

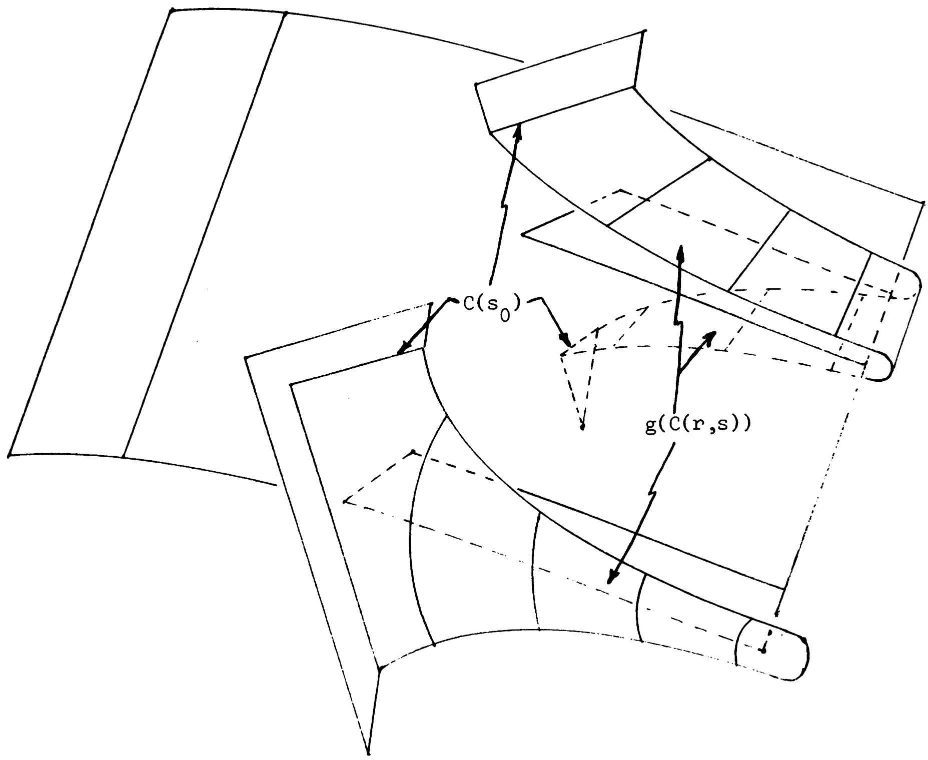

Figure 4

Proposition 4.3. *Suppose that K is a 2-complex in E^m such that $K \cap D[0,a] = \varnothing$. Then there is a neighborhood W of $D[0,a]$ such that for every $\epsilon > 0$ there exists an isotopy g_t $(t \in I)$ of E^m satisfying*:

(i) $g_0 =$ *identity*,

(ii) $g_t \mid W \cup (E^m - N_\epsilon(D[a,b])) =$ *identity for all* $t \in I$,

(iii) $g_1(K) \cap D[0,b] = \varnothing$, *and*

(iv) *for each* $y \in E^m$ *either* $g_t(y) = y$ *for all* $t \in I$ *or there exists* $x \in I^{k-1}$ *such that* $g_t(y) \in N_\epsilon(f(x \times [a,b]))$ *for all* $t \in I$.

Proof. The proof of this lemma uses the homotopy provided by Lemma 4.2 together with the radial engulfing technique of [3], [13], and [16]. In particular, one must use Zeeman's piping lemma [17] in case $m = 5$ as in [16]. To get the neighborhood W of $D[0,a]$, first select $c > a$ $(c \leqslant b)$ so that $K \cap D[0,c] = \varnothing$, and then simply choose W so that $\overline{W} \cap D[0,b] \subset D[0,c]$.

Theorem 4.4. *Suppose that W is a neighborhood of $D[0,a]$ and $\epsilon > 0$. Then there exist a neighborhood W' of $D[0,a]$ and an isotopy h_t $(t \in I)$ of E^m such that*

(i) $h_0 =$ *identity*,

(ii) $h_t \mid W' \cup (E^m - N_\epsilon(D[a,b])) =$ *identity*,

(iii) $h_1(W) \supset D[0,b]$, *and*

(iv) *if $y \in E^m$, either $h_t(y) = y$ for all $t \in I$ or there exists $x \in I^{k-1}$ such that $g_t(y) \in N_\epsilon(f(x \times [a,b]))$ for all $t \in I$.*

Proof. Suppose $\delta > 0$. Choose W'' and $\gamma > 0$ corresponding to W and δ as in Proposition 4.1. Let M be a PL neighborhood of $D[a,b]$ lying in $N_\gamma(D[a,b])$, and let T be a fine triangulation of M with the property that no simplex of T can intersect both $D[0,a]$ and $E^m - W''$. Let K be the subcomplex of T^2, the 2-skeleton of T, obtained by taking all $\sigma \in T^2$ such that $\sigma \cap D[0,a] = \varnothing$, and let L be the dual of K in T (i.e., L consists of all simplexes σ in T', the first barycentric subdivision of T, such that $\sigma \cap K = \varnothing$). Then $\dim(L - W'') \leqslant m - 3$, and hence $L - W''$ lies in a subcomplex L_1 of L such that $\dim L_1 \leqslant m - 3$. Let W' be a neighborhood of $D[0,a]$ corresponding to K as in Proposition 4.3 and having the additional property that $W' \cap M \subset \operatorname{Int} L$ (interior relative to M) and $W' \subset W''$.

Let h'_t $(t \in I)$ be the isotopy of E^m satisfying (i)–(iv) of Proposition 4.1 with $\{\delta, L_1, W''\}$ replacing $\{\epsilon, P, W'\}$. Since $h'_t \mid W'' =$ identity for all $t \in I$, $h'_1(W) \supset L$.

Now let g_t $(t \in I)$ be an isotopy of E^m satisfying (i)–(iv) of Proposition 4.3 with $\{\lambda, K, W'\}$ replacing $\{\epsilon, K, W\}$, where $\lambda > 0$ is small enough so that $\lambda \leqslant \delta$ and $N_\lambda(D[a,b]) \subset M$. Then $g_t \mid W' \cup (E^m - M) =$ identity for all $t \in I$.

We now have $h'_t(W) \supset L$ and $g_1^{-1}(E^m - D[0,b]) \supset K$, where K' and L are dual subcomplexes of T'. Hence, there is an isotopy ϕ_t $(t \in I)$ of E^m that is fixed outside a neighborhood of M and on W' such that $\phi_0 =$ identity and $\phi_1 h'_1(W) \cup g_1^{-1}(E^m - D[0,b]) \supset M$. (This is Stallings' isotopy [14].) Moreover, the distance a point moves under ϕ_t is no greater than the mesh of T (hence, arbitrarily small). Observe that

$$g_1 \phi_1 h'_1(W) \cup (E^m - D[0,b]) \supset g_1(M) = M$$

and of each g_t, ϕ_t, and h'_t $(t \in I)$ is fixed on W' so that

$$g_1 \phi_1 h'_1(W) \supset D[0,b].$$

If δ is sufficiently small, then the desired isotopy h_t $(t \in I)$ of E^m is obtained by "stacking" the isotopies h'_t, ϕ_t, and g_t (in that order).

5. **The proof of Theorem 1.1.** In this section we shall set up the machinery so that we can appeal to the methods of [2] and [4]. As before, Y denotes a space with the property that $Y \times E^1 \simeq E^{n+1}$.

Theorem 5.1. *If $f: I^{k-1} \times I \to Y$ ($n \geqslant 4$) is an embedding with $D = f(I^k)$, then for each $\epsilon > 0$ there exists an isotopy h_t ($t \in I$) of E^{n+1} satisfying*:

(1) $h_0 =$ *identity*,

(2) $h_t =$ *identity outside* $N_\epsilon(D \times E^1)$,

(3) h_t *is uniformly continuous*,

(4) *for each* $z \in E^{n+1}$, *either* $h_t(z) = z$ *for all* $t \in I$ *or there exist* $x \in I^{k-1}$ *and* $w \in E^1$ *such that* $h_t(z) \in N_\epsilon(f(x \times I) \times w)$ *for all* $t \in I$, *and*

(5) *for all* $w \in E^1$ *there exists* $y \in I$ *such that for all* $x \in I^{k-1}$,

$$h_1(f(x \times I) \times w) \subset N_\epsilon(f(x,y) \times w).$$

Theorem 5.1 is Statement $H(n,k,1)$ of [4] with $n \geqslant 4$. Once we have proved this theorem, we will be through because we proved in [4] that $H(n,k,1)$ implies Theorem 1.1. (See Lemmas 2.1 and 2.2 of [4].)

Proof of Theorem 5.1. Suppose we are given $f: I^{k-1} \times I \to Y$ ($n \geqslant 4$) and $\epsilon > 0$. As usual we shall set $D = f(I^k)$ and $D[a,b] = f(I^{k-1} \times [a,b])$.

Let $a_0 = 0 < a_1 < a_2 < \ldots < a_m = 1$ be numbers in I, and choose δ ($0 < \delta < \epsilon$) so that $N_\delta(D[0,a_{i-1}] \times E^1) \cap N_\delta(D[a_i,1]) = \varnothing$ for $i = 1, \ldots, m - 1$.

Let ϵ_1 be a positive number and let N_1 be a neighborhood of $D[0,a_{m-1}]$ such that $\overline{N}_1 \subset N_\epsilon(D) \cap (Y \times (-\epsilon_1, \epsilon_1))$. Apply Theorem 4.4 and get an isotopy h_t^1 ($t \in I$) such that

$h_0^1 =$ identity,

$h_t^1 =$ identity outside $N_\delta(D[a_{m-1}, 1] \cap (Y \times (-\epsilon_1, \epsilon_1))$,

$h_1^1(N_1) \supset D$, and

h_t^1 satisfies condition (iv) of Theorem 4.4 with $\{a_{m-1}\, 1, \epsilon/2\}$ replacing $\{a, b, \epsilon\}$.

There exists ϵ_2 ($0 < \epsilon_2 < \epsilon_1$) such that $h_1^1(N_1) \supset D \times [-\epsilon_2, \epsilon_2]$. Let N_2 be a neighborhood of $D[0,a_{m-2}]$ such that

$$\overline{N}_2 \subset N_\delta(D[0,a_{m-2}]) \cap (Y \times (-\epsilon_2, \epsilon_2)).$$

Let λ_2 be a positive number, and apply Theorem 4.4 to get an isotopy h_t^2 ($t \in I$) of E^{n+1} such that

$h_0^2 =$ identity,

$h_t^2 =$ identity outside $N_\delta(D[a_{m-2}, 1]) \cap (Y \times (-\epsilon_2, \epsilon_2))$,

$h_1^2(N_2) \supset D$, and

h_t^2 satisfies (iv) of Theorem 4.4 with $\{a_{m-2}, 1, \lambda_2\}$ replacing $\{a, b, \epsilon\}$.

We continue in this manner, obtaining numbers $\epsilon_1 > \epsilon_2 > \ldots > \epsilon_m > 0$, neighborhoods N_i of $D[0,a_{m-i}]$ ($i = 1, 2, \ldots, m - 1$), positive numbers $\lambda_2, \lambda_3, \ldots, \lambda_{m-1}$ and isotopies h_t^i ($t \in I$) ($i = 1, 2, \ldots, m - 1$) that satisfy

h_0^i = identity,

h_t^i = identity outside $N_\delta(D[a_{m-i}, 1]) \cap (Y \times (-\epsilon_i, \epsilon_i))$,

$h_1^i(N_i) \supset D \times [-\epsilon_{i+1}, \epsilon_{i+1}]$,

h_t^i satisfies (iv) of Theorem 4.4 with $\{a_{m-i}, 1, \lambda_i\}$ replacing $\{a, b, \epsilon\}$,

where $\lambda_1 = \epsilon/2$.

Observe that $h_t^j \mid N_i$ = identity if $i > j$. If the numbers $\lambda_2, \lambda_3, \ldots, \lambda_{m-1}$ are chosen properly, then the homeomorphism $h = (h_1^1)^{-1}(h_1^2)^{-1} \ldots (h_1^{m-1})^{-1}$ will be the 1-level of an isotopy h_t $(t \in I)$ having the following properties:

$h(D \times [-\epsilon_{i+1}, \epsilon_{i+1}]) \subset N_i$ $(i = 1, \ldots, m-1)$,

h_t = identity outside $N_\delta(D[a_{m-i}, 1]) \cap (Y \times (-\epsilon_i, \epsilon_i))$, and

h_t satisfies (iv) of Theorem 4.4 with $\{0, 1, \epsilon\}$ replacing $\{a, b, \epsilon\}$.

We may now appeal to the technique of proof in Lemma 2 of [2] to complete the proof of Theorem 5.1. (See also Theorems 4.2 and 4.3 of [4].)

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